A NEW METHOD FOR SOLUTION OF A CLASS OF PROBLEMS IN ANALYTICAL GEOMETRY

By
A. R. Khalifa*
Lecturer in Mathematics, Dacca University

A very easy, simple and interesting method for solution of a class of problems in Analytical Geometry in connection with equations of circles and spheres is given below.

If $S = 0$ and $S' = 0$ are equations of two curves, then $S + KS' = 0$ is the general equation of all curves, passing through all the points common to the two curves, In the particular case when $S = 0$ is the equation of a circle, and $L = 0$ is the equation of a straight line, then $S + KL = 0$, or, what amounts to the same thing,

$$S = AL \tag{1}$$

is the general equation of all circles passing through the two points of intersection of the circle $S = 0$ with the straight line $L = 0$.

Proposition I. To find the general equation of all circles passing through two given points (x_1, y_1) and (x_2, y_2).

In consideration of the fact that the equation of a circle must be of the second degree, in which the coefficients of x^2 and y^2 must be equal, and the term containing xy must be absent (for rectangular co-ordinates) and in order that the circle may pass through the two points, the coordinates of the 2 points must satisfy it, the equation of a circle passing through the given points (x_1, y_1) and (x_2, y_2) may be written as

$$S \equiv (x - x_1)\,(x - x_2) + (y - y_1)\,(y - y_2) = 0. \tag{2}$$

It may be seen in passing that this equation is obtained in Analytical Conics in another context, namely as the equation of the circle drawn upon the straight line joining the two given points as its diameter. We do not require that property. This may only be noted how easily the equation can be written down at once from above considerations.

* মৃত্যুবরণ: ঢাকা ডিসেম্বর ১৯৮৪।

The equation of a straight line through the two points (x_1, y_1) and (x_2, y_2) is written very easily from similar considerations in the same way as

$$L \equiv (x - x_1)\,(y - y_2) - (x - x_2)\,(y - y_1) = 0.$$

Therefore the general equation of all circles passing through the two points (x_1, y_1) and (x_2, y_2) is from (1),

$$(x - x_1)\,(x - x_2) + (y - y_1)\,(y - y_2)$$
$$= A\,(x - x_1)\,(y - y_2) - (x - x_2)\,(y - y_1) \tag{3}$$

Now we work out some problems with the help of this result.

Example 1. Find the equation of the circle passing through the three points

$$(-3, 2), (1, 7) \text{ and } (5, -3).$$

The general equation of all circles passing through the first two points $(-3, 2)$ and $(1, 7)$ is

$$(x + 3)\,(x - 1) + (y - 2)\,(y - 7)$$
$$= \mathrm{A}\,\{(x + 3)\,(y - 7) - (x - 1)\,(y - 2)\}. \tag{4}$$

Since the circle passes through the third point (5,-3), we have

$8.4 + (-5)\,(-10) = A\{8(-10) - 4(-5)\}$; *i.e.*, $82 = A(-60)$;

giving $41 = -30\,A$ (5)

By division of (4) by (5), cross-wise multiplication, and by simplification orally. we get

$$30\,(x^2 + y^2 + 2x - 9 + 11) = -41\,(-5x + 4y - 23);$$

i.e. $30\,(x^2 + y^2) + (60 - 205)x + (164 - 270)y + (330 - 943) = 0$

i.e. $30\,(x^2 + y^2) - 145x - 106y - 613 = 0$;

which is the required equation.

Example 2. Find the equation of the circle which passes through the points $(1, -1)$ and $(2, 3)$ and has its centre on the straight line $x + 2y = 1$.

The general equation of a circle passing through the points $(1, -1)$ and $(2, 3)$ is

$$(x-1)\,(x-2) + (y+1)\,(y-3)$$
$$= A\,\{(x-1)\,(y-3) - (x-2)\,(y+1)\}; \qquad (6)$$

i.e. $x^2 + y^2 - 3x - 2y - 1 = A\,(-4x + y + 5)$;

i.e. $x^2 + y^2 + (4A-3)x - (2+A)y - 1 - 5A = 0$; (7)

The centre of this circle, namely $\left(-\frac{4A-3}{2}, \frac{2+A}{2}\right)$, lies on the line $x + 2y - 1 = 0$; . so that

$$-\frac{4A-3}{2} + 2\frac{2+A}{2} - 1 = 0, \textit{ i.e. } -4A + 3 + 4 + 2A - 2 = 0.$$

i.e. $2A = 5$, or $A = 5/2$.

Putting this value of A in (7),

$$(x^2 + y^2) + 7x - \frac{9}{2}y - \frac{27}{2} = 0\,;$$

i.e. $2(x^2 + y^2) + 14x - 9y - 27 = 0$

which is the required equation.

Example 3. Find the equation of the circle tangent to the line $2y = 3x$ at (2, 3) and passing through (4, 5) – *Ex.* (illustration page 112, *College Mathematics* by Dr. Kaj. L. Nielson, Barnes & Noble, New York).

The equation of a circle passing through the 2 points (2, 3) and (4, 5) is
$(x-2)\,(x-4) + (y-3)\,(y-5) = A\,\{(x-2)\,(y-5) - (x-4)\,(y-3)\}$ (8)

i.e. $x^2 + y^2 - 6x - 8y + 23 = A\,(-2x + 2y - 2)$;

i.e. $x^2 + y^2 + 2\,(A-3)\,x - 2\,(A+4)\,y + 23 + 2A = 0$ (9)

The equation of the tangent at (x^1, y^1) to this circle is

$$xx^1 + yy^1 + (A-3)\,(x + x^1) - (A+4)\,(y + y^1) + 23 + 2A = 0.$$

The tangent at (2, 3) is therefore

$$2x + 3y + (A-3)\,(x+2) - (A+4)\,(y+3) + 23 + 2A = 0. \qquad (10)$$

This must be same as $2y = 3x$, and this equation has no constant term;

i.e. constant term of (10) must be *zero*, so that

$$2(A-3) - 3\,(A+4) + 23 + 2A = 0;\ \text{i.e. } A + 5 = 0, \text{ or } A = -5.$$

Putting this value of A in (9), we get, $x^2 + y^2 - 16x + 2y + 13 = 0$, as the required equation.

Example 4. Find the equation of the circle passing through the points (x', y'), (x'', y'') and (x''', y''').

The general equation of the circle passing through the two points (x', y') and (x'', y'') is

$$(x-x')\,(x-x'') + (y-y')\,(y-y'')$$
$$= A\{(x-x')\,(y-y'') - (x-x'')\,(y-y')\} \qquad (11)$$

Since this passes through (x''', y'''), we have

$$(x'''-x')\,(x'''-x'') + (y'''-y')$$
$$= A\{(x'''-x')\,(y'''-y'') - (x'''-x'')\,(y'''-y')\}; \qquad (12)$$

Dividing (11) by (12) we have for the required equation,

$$\frac{(x-x')(x-x'')+(y-y')\,y'')}{(x'''-x')(x'''-x'')+(y'''-y')(y'''-y'')}$$
$$= \frac{(x-x')(y-y''-(x-x'')(y-y')}{(x'''-x')(y'''-y')-(x'''-x'')(y'''-y')}$$

This form of the equation is as compact as that given under the determinant form by the usual methods given in the text books; but this is more suitable for numerical examples. It is however, better, to work out numerical examples independently, as we have done in Ex. l above.

The method is readily generalised for three dimensional geometry. If $S = 0$, be the equation of a sphere, and $L = 0$ is the equation of a plane, then the general equation of all the spheres passing through their curve of intersection is

$$S = AL \qquad (13)$$

Proposition 2. Find the equation of the sphere passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).

The equation of a sphere passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is easily written as before.

$$S \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0. \qquad (14)$$

The equation of a plane passing through these two points may be written in the form

$$L \equiv (x - x_1)(y - y_2) - (x - x_2)(y - y_1) + R(x - x_1)(z - z_2) \qquad (15)$$

It may easily be verified that (15) is the equation of a plane passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2,); because this equation is satisfied by the coordinates for all values of R; and further in (15) the terms xy and xz cancel out, so that (15) is an equation of the first degree in x, y and z and as such it is the equation of a plane.

Putting the expressions for S and L in (13), we get the general equation of a sphere passing through two points. The equation of the sphere can be obtained by eliminating the constant A and R from the fact that the sphere passes through the other 2 points. The procedure can be gathered from the following example.

Example 1. Find the equation of the sphere passing through the points (2, 1, 3,), (1, 4, 2), (5, – 1, 4) and (6, 2, –1).

The general equation of a sphere passing through two given points (2, 1, 3) and (1, 4, 2) is from above

$(x - 2)(x - 1) + (y - 1)(y - 4) + (z - 3)(z - 2)$

$= A\,[\{(x - 2)(y - 1)(y - 1) + R\{(x - 2)(z - 2) - (x - 1)(z - 3)\}]$ (16)

If this sphere passes through (5, – 1, 4) we get

$3.4 + (-2)(-5) + 1, 2 = A\{3(-5) - 4(-2) + R(3.2 - 4, 1)\}$;

i.e. $-7A + AR - 24 = 0,$ (17)

The point (6, 2, – 1) lying on (16),

$4.5 + 1(-2) + (-4)(-3) = A[\{4(-2) - 5.1) + R\{4(-5(-4)\}]$

i.e. $13A - 8AR + 30 = 0.$ (18)

Writing (17) and (18) as

$$7A - (2AR) + 4.6 = 0 \text{ and } 13A - 4(2AR) + 5.6 = 0.$$

and then by cross-multiplication, $\dfrac{A}{-5+16} = \dfrac{2\,AR}{52-35} = \dfrac{6}{-28+13}$;

i.e. $\dfrac{A}{11} = \dfrac{2\,AR}{17} = \dfrac{6}{-15} = \dfrac{-2}{5}$

giving $A = -\dfrac{22}{5}$ and $AR = -\dfrac{17}{5}$

Putting the values of A and AR in (16), we get on simplifying,

$5(x^2 + y^2 z^2) - 3x - 5y - 5z + 12 = -22(-3x - y + 7) - 17(x - z + 1),$

i.e. $5(x^2 + y^2 + z^2) - (15 + 66 - 17)x - (25 + 22)y - 2(25 + 22)y - 2(25 + 17)z + 60 + 154 + 17 = 0$;

i.e. $5(x^2 + y^2 + z^2) - 64x - 47z + 231 = 0,$

which is the required equation.

As another illustration of the method we work out the following example from page 120 of the same book: *College Mathematics* by Dr. Nielson:

Find the equation of the parabola with its axis parallel to OY and passing through (1, 1), (0, 5) and (2, 5).

The equation of a parabola with its axis parallel to y-axis is of the form $(x - h)^2 = A(y - k)$, which is same as $x^2 = AL$, (18 A)

where $L = 0$ is the equation of a straight line.

We take $L = 0$, as the equation of the straight line passing through the points (1,1) and (0,5), which is $(x - 1)(y - 5) - (x - 0)(y - 1) = 0$;

or $-4x - y + 5 = 0$; *or* $L - 4x + y - 5 = 0$.

The equation of a parabola passing through the points (1, 1) and (0, 5) can be written as $(x - 1)(x - 0) = A(4x + y - 5)$.

This passes through the third point (2, 5), so that we get

$$(2 - 1)\,2 = A(4.2 + 5 - 5); \text{ or } 2 = 8A, \text{ giving } A = 1/4.$$

Substituting this value of A in (19), we get

$$4(x^2 - x) - (4x + y - 5) = 0, \text{ or } 4x^2 - 8x - y + 5 = 0,$$

for the required equation.

Many examples of circles passing through two given points and with another condition attached can be solved very easily and at a much shorter time by this method, Equations of spheres passing through two given points and with two other given conditions may similarly be tackled. Cases of parabola and other conics passing through two points and with other conditions yield to the method under special circumstances.

Equation (15) for the plane passing through two given points may be very helpful, and is a very easy and convenient method for finding the equation of a plane passing through 3 given points, or through two given points and under one more given condition, at least in numerical cases. The point to note is the case with which it is written, from the mere consideration that an equation of a plane must be of the first degree in the variables *x*, *y* and *z*, and that the coordinates of the two given points must satisfy the equation.

We illustrate by some examples.

Example 1. Find the equation of the plane passing through the points

(2, 1, –3), (3, – 1, 4) and (7, 5, 6).

The general equation of a plane passing through the first two points

(2, 1, – 3), and (3, – 1, 4) is

$$(x-2)(y+1)-(x-3)(y-1)=A\{(x-2)(z-4)-(x-3)(x+3)\} \quad (20)$$

This passes through the 3^{rd} point (7, 5, 6) so that,

$$5.6-4.4=A\,(5.2-4.9),\ i.e.\ 14=-26A;\ or\ A=-\frac{7}{13}.$$

Putting this value in (20); and simplifying simultaneously,

$13(2x+y-5)+7(-7x+z+17)=0,$

i.e. $-23x+13y+7z+(119-65)=0;$

i.e. $23x-13y-7z-54=0,$

which is the required equation.

Example 2. Find the equation of the plane through the line

$$\frac{x-2}{3}=\frac{y-3}{5}=\frac{z}{7} \quad (21)$$

and passing through the point (1, – 2, 3).

(Illustrative Example, page 34, Co-ordinate Geometry with vectors and Tensors by E. A. Maxwell, Oxford University Press, 1958)

(2, 3, 0) and (5, 8, 7) are evidently two points of the line (21).

[Any number of points of the line can be obtained by putting the equations (21) equal to a constant, say *t*, and giving to *t*, numerical values at will, the two points above correspond to the values 0 and 2 of *t*.]

We may then proceed to find the equation of the plane through the 3 points (2, 3, 0), (5, 8, 7) and (1, – 2, 3), in the above manner.

But the best way to tackle this present problem by the present method, would probably be, as follows:

The general equation of the plane containing the line (21) is

$$5(x-2)-3(y-3)=K\{7(x-2)-3z\} \quad (22)$$

[It may be seen that this equation is obtained by splitting up (21) into two equations, then multiplying cross-wise and connecting the result by the variable constant *K* for the general case, the value of *K* to be fixed by the additional condition.]

Since this plane passes through (1, – 2, 3), we have

$$5(-1)-3(-5)=K\{7(-1)-9\}\ i.e.\ 10=-16K;\ or\ K=\frac{-5}{8}.$$

Putting this value of *K* in (22),

$8(5x-3y-1)=-5(7x-3z-14);$ *i.e.* $75x-24y-15z-78=0;$

i.e. $25x-8y-5z-26=0.$

is the required equation.

I think we should take up another example to illustrate what combinations of *x*, *y*; *y*, *z*; *z*, *x* to choose, and also to point out a very trivial case.

In (15) in the left hand side we chose the combination of *x*, *y* and on the right, that of *x* and *z*. In the right we might instead choose the combination of *y* and *z* as well. There may be cases when right hand side becomes zero, after putting the values of the coordinates of the third point, and the value of the constant cannot be found. So the choice of the combination of *x*, *y*, *z* may properly be made to avoid the trouble. I take up the following more interesting case.

Example 3. Determine the plane through the line whose parametric equations are $x = -4 + 3t$, $y = 5 - t$, $z = 3 + 2t$ and through the point (– 4, 3, 3). (Worked out Ex. 2, page 93, Solid Analytical Geometry and determinants, by Arnold Dresden, Ph.D.; John Wiley & Sons, Inc., New York.)

This is in fact the same problem as *Ex.* 2 worked above, but a bit easier. It is not difficult to prove that the equation of the plane containing the line

$$\frac{x-a}{l}=\frac{y-b}{m}\frac{z-c}{n} \qquad (22)$$

and the point (p, b, c) is

$$\frac{y-b}{m}=\frac{z-c}{n}$$

Similarly the equation of the plane containing the line (22) and passing through the point (a, q, c) is

$$\frac{x-a}{l}=\frac{z-c}{n};$$

and the equation of the plane passing through the line (22) and the point (a, b, r) is similarly

$$\frac{x-a}{l}=\frac{y-b}{m}$$

In view of these, the answer comes immediately by eliminating *t* from the values of *x* and *z*, as

$$\frac{x+4}{3}=\frac{z-3}{2} \text{ i.e. } 2x-3z+17=0.$$

which is the required equation.

[Whenever I get any opportunity to set questions for examinations, or conduct a *viva-voce* examination, I put some questions which ordinary students work out by ordinary (classical) methods, whereas sometimes intelligent students can answer them in a matter of mere one or two steps. The above example is an instance of one such question. Such matters have been described in detail in my articles

(i) Construction of new examples in *Trigonometry*, *Hydrostatics*, *Analytical Geometry*;

(ii) Choice of examples in textbooks; and

(iii) Questions to set in Examinations to encourage students to their work, published elsewhere.]

A. R. Khalifa (আজিজর রহমান খলিফা : ১৯০৪-৮৪)। স্বনামখ্যাত গণিতবিদ। কোলকাতা বিশ্ববিদ্যালয়ের ১৯২৭-এর গণিতে এম.এ.। দীর্ঘদিন (১৯৪৮-৬৭, ১৯৭৩-৭৫) ঢাকা বিশ্ববিদ্যালয়ে অধ্যাপনা করেছেন। নিবেদিতপ্রাণ একজন শিক্ষক হিসেবে প্রশংসিত হয়েছেন। ‘গণিত কাঁদিয়া ফেরে’ তার একটি অপ্রকাশিত মূল্যবান রচনা। *The Dacca University Studies*, Part II, Vol. XI, June 1963, pp. 73-79 প্রকাশিত তার এই পেপার খুবই গুরুত্ব বহন করে বিধায় একশ্রেণির পাঠকের একান্ড় অনুরোধে পরিক্রমার জন্য তা পুনর্মুদ্রণ করা হলো। - **m¤úv`K**